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Integrable Limits of Dynamics in Trapped Bose-Condensates

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The dynamics of quasiparticles in Bose condensates at zero temperature, confined in harmonic potentials, are studied using the Bogoliubov-theory. The Hamiltonian of the Bogoliubov-theory, appearing in the semiclassical limit is investigated in detail. The classical motion given by this Hamiltonian is generally chaotic already for axially symmetric traps. But, in certain parameter regions the motion becomes quasi-integrable. Integrable regions are studied classically, and the experimentally accessible low-energy region quantum mechanically.

KEY WORDS: Bose–Einstein condensation in traps; semiclassical-limit; integrable-problems.

1. INTRODUCTION AND SUMMARY

Since the first realizations of Bose–Einstein condensates in 1995,[1, 2, 3] with Rb, Li, Na there are already more then twenty experimental groups who are able to cool down atoms in magnetic traps below the critical temperature of Bose–Einstein condensation. Several effects have been studied experimentally using condensates (see for experimental reviews[4, 5]), which initiated an immense number of theoretical works (for theoretical reviews see refs. 6 and 7). The common features of the nowadays experiments are, that atoms are confined in harmonic trap potential and the atom-cloud is very dilute. These facts call for a space-dependent version of Bogoliubov-theory at zero temperature, which is applicable due to the diluteness. One

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must go beyond Bogoliubov-theory for larger temperature, where the ratio of the noncondensed atoms to the total number of atoms is non-negligible, or if one is interested in dampings of elementary excitations.

The many body Hamiltonian, which describes the interacting atoms trapped in the external potential $U(\mathbf{r})$ is given in second quantized form by

$$\hat{H}-\mu\hat{N}=\int d^3r\ \hat{\Psi}^+(\mathbf{r})\left(-\frac{\hbar^2}{2m}\Delta+U(\mathbf{r})-\mu\right)\hat{\Psi}(\mathbf{r})$$

$$+\int d^3r\int d^3r'\ \hat{\Psi}^+(\mathbf{r})\ \hat{\Psi}^+(\mathbf{r}')\ v(\mathbf{r}-\mathbf{r}')\ \hat{\Psi}(\mathbf{r}')\ \hat{\Psi}(\mathbf{r}) \tag{1}$$

where the field operator follows Bose-statistics: $[\hat{\Psi}(\mathbf{r}), \hat{\Psi}^+(\mathbf{r}')]=\delta(\mathbf{r}-\mathbf{r}')$. In the recent experiments the trap potential is chosen to be a harmonic oscillator one

$$U(\mathbf{r})=\tfrac{1}{2}m\omega_x^2x^2+\tfrac{1}{2}m\omega_y^2y^2+\tfrac{1}{2}m\omega_z^2z^2 \tag{2}$$

In most of the experiments the trap potential is axially symmetric ($\omega_x=\omega_y$) but, the case of the triaxially anisotropic harmonic trap ($\omega_x\neq\omega_y\neq\omega_z$) has also been realized.[8] Other characteristics of the new experiments are that the achieved temperatures are rather small and the density of the trapped atoms are extremely small. Consequently, at the occuring temperatures and densities the two-body interaction can be treated in the s-wave approximation:

$$v(\mathbf{r}-\mathbf{r}')=\frac{4\pi\hbar^2a_0}{m}\,\delta(\mathbf{r}-\mathbf{r}')\equiv g\delta(\mathbf{r}-\mathbf{r}'), \tag{3}$$

where a_0 is the s-wave scattering length, which is here assumed to be positive (Up to now, there is only one experiment of successfull Bose–Einstein condensation[2] using ^{7}Li, where a_0 was negative).

Our aim is here to investigate the quasi particle dynamics for Bose condensates of atomic gases in traps given by the Bogoliubov-theory. It has been found in refs. 9 and 10 that the classical dynamics of quasiparticles is generally nonintegrable already for axially symmetric harmonic traps ($\omega_x=\omega_y=\omega_0$). Chaos shows up most dominantly at intermediate energies $E\approx\mu$, where μ is the chemical potential, and has a direct consequence for quantum dynamics, because it implies avoided level crossings between quasiparticle levels changing some external parameter (for example the ratio ω_0/ω_z). We shall find however, some quasi-integrable parameter regions. It will be shown that the motion is asymptotically integrable for high energies, where the approximate classsical Hamiltonian is the Hartree–Fock Hamiltonian. At low energies there is one region, where the motion is given approximatelly by the Hartree–Fock Hamiltonian as well, which is also quasi-integrable. Finally, we shall show that the low-lying collective modes, first discussed theoretically by Stringari,[11] becomes integrable classically and the corresponding Schrödinger equation is separable in elliptic coordinates. As a new result we shall calculate the full spectra of collective modes for highly deformed traps in two limiting cases: $\omega_0 \gg \omega_z$ and $\omega_z \gg \omega_0$.

The paper is organized as follows. Section 2 is devoted to the derivation of space-dependent Bogoliubov-theory for quasiparticles and to its semiclassical limit. In Section 3 we shall investigate the classical behavior of the corresponding Hamiltonian for different parameters. In Section 4 we shall discuss the quasi-integrable regimes classically and derive conserved quantities ensuring integrability in these regimes. In Section 5 we shall solve the quantum mechanical problem of low lying excitations. As a new result we derive the full quasiparticle spectra for highly deformed harmonic traps. Section 6 is devoted for final remarks.

2. DERIVATION OF THE SEMICLASSICAL LIMIT OF THE BOGOLIUBOV-EQUATIONS

The zero-temperature, mean-field theory starts by splitting the field operator into a C-number part $\Phi_0(\mathbf{r})$ (the condensate wave function) and a residual operator $\hat{\Phi}(\mathbf{r})$: $\hat{\Psi}(\mathbf{r}) = \Phi_0(\mathbf{r}) + \hat{\Phi}(\mathbf{r})$ and the decomposition of (1) in terms of different order in $\hat{\Phi}$, $\hat{\Phi}^+$. In the Bogoliubov-approximation the terms of order 3 and 4 are neglected. The term of order one is made to vanish by choosing $\Phi_0(\mathbf{r})$ to satisfy the Gross–Pitaevskii equation[12]

$$\left(-\frac{\hbar^2}{2m}\Delta + U(\mathbf{r}) - \mu + g\,|\Phi_0(\mathbf{r})|^2\right)\Phi_0(\mathbf{r}) = 0 \tag{4}$$

It must be solved with the normalization

$$N_0 = \int d^3r\,|\Phi_0(\mathbf{r})|^2 \tag{5}$$

where according to the assumptions of the Bogoliubov-theory all atoms are supposed to be in the condensate: $N_0 \approx N$.

For $N_0 a_0/d \gg 1$, where $d = \sqrt{\hbar/m\bar{\omega}}$ is the characteristic oscillator length and $\bar{\omega} = (\omega_x \omega_y \omega_z)^{1/3}$ the solution to the Gross–Pitaevskii equation can be well approximated by the Thomas–Fermi approximation[13] which neglects the kinetic-energy term in (4):

$$|\Phi_0(\mathbf{r})|^2 = \frac{1}{g}(\mu - U(\mathbf{r}))\,\Theta(\mu - U(\mathbf{r})) \tag{6}$$

According to the experimental data $N_0 a_0/d \gg 1$ is fullfilled in most of the experiments sufficiently below T_c.

The next step of the zero-temperature mean-field procedure is the diagonalization of that part of (1) which is of order 2 by a space dependent Bogoliubov-transformation $\hat{\Phi}(\mathbf{r}) = \sum_j' [u_j(\mathbf{r})\,\hat{\alpha}_j - v(\mathbf{r})\,\hat{\alpha}_j^+]$ to quasiparticles. The quasiparticle operators $\hat{\alpha}_i$ fullfill the standard Bose-commutator relations $[\hat{\alpha}_i, \hat{\alpha}_j^+] = \sigma_{ij}$. The order 2 part of $\hat{H} - \mu\hat{N}$ is diagonalized if the space-dependent coefficients satisfy the coupled Bogoliubov-equations[14]

$$\begin{pmatrix} \hat{H}_{HF} & -g\Phi_0^2(\mathbf{r}) \\ -g\Phi_0^{*2}(\mathbf{r}) & \hat{H}_{HF} \end{pmatrix} \begin{pmatrix} u_j(\mathbf{r}) \\ v_j(\mathbf{r}) \end{pmatrix} = E_j \begin{pmatrix} u_j(\mathbf{r}) \\ -v_j(\mathbf{r}) \end{pmatrix} \tag{7}$$

with the Hartree–Fock Halnilton-operator

$$\hat{H}_{HF} = -\frac{\hbar^2}{2m}\Delta + U(\mathbf{r}) - \mu + 2g\,|\Phi_0(\mathbf{r})|^2 \tag{8}$$

In order to study the (possibly) chaotic behavior one must apply the semiclassical limit of Eq. (7), in which process one can obtain the corresponding classical Hamilton–Jacobi equation of the problem. The easiest way to arrive at Hamilton–Jacobi equation is to apply the semiclassical ansatz

$$\begin{pmatrix} u_j(\mathbf{r}) \\ v_j(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} u_j^0(\mathbf{r}) + o(\hbar) \\ v_j^0(\mathbf{r}) + o(\hbar) \end{pmatrix} e^{i(S(\mathbf{r}) + o(\hbar))/\hbar} + \cdots \tag{9}$$

to Eq. (7).[9] In zeroth order in $\hbar$ (7) reduces to

$$\begin{pmatrix} \varepsilon_{HF} - E & -g\Phi_0(\mathbf{r})^2 \\ -g\Phi_0(\mathbf{r})^{*2} & \varepsilon_{HF} + E \end{pmatrix} \begin{pmatrix} u_j^0(\mathbf{r}) \\ v_j^0(\mathbf{r}) \end{pmatrix} = 0 \tag{10}$$

with $E \equiv E_j$, $\mathbf{p} = \partial S/\partial \mathbf{r}$ and

$$\varepsilon_{HF}(\mathbf{p}, \mathbf{r}) = \frac{p^2}{2m} + U(\mathbf{r}) - \mu + 2g\,|\Phi_0(\mathbf{r})|^2 \tag{11}$$

The zeroth-order equations are homogeneous first order linear equations for the coefficients u_j^0 and v_j^0 and correspondingly, have nontrivial solution only, if the determinant condition $E^2 = \varepsilon_{HF}^2(\mathbf{p}, \mathbf{r}) - g^2|\Phi_0(\mathbf{r})|^4$ is satisfied, which gives the time-independent Hamilton–Jacobi equation $E = H(\partial S/\partial \mathbf{r}, \mathbf{r})$ with the classical Hamiltonian

$$H(\mathbf{p}, \mathbf{r}) = \sqrt{\varepsilon_{HF}^2 - g^2 \, |\Phi_0(\mathbf{r})|^4} \tag{12}$$

In the following we analyze the classical Hamiltonian (12) of the Bogoliubov quasiparticles.

3. CLASSICAL BEHAVIOR OF BOGOLIUBOV QUASIPARTICLES

For the case of isotropic harmonic traps the angular momentum vector is conserved, thus the quasiparticle dynamics is integrable. This classical integrability is reflected on quantummechanical level by the fact that the Bogoliubov-equations are separable in spherical coordinates. Therefore, in this section we concentrate on axially symmetric harmonic traps ($\omega_x = \omega_y \neq \omega_z$) with the condensate taken in Thomas–Fermi approximation. In this case conserved quantities are the z-component of the angular momentum (L_z) and the energy (E) itself. For an integrable 3 degree of freedom problem one needs a third conserved quantity. The question naturally arises: does a third conserved quantity exist for axially symmetric traps or not?

A detailed numerical study of the classical motion governed by (12) can be found in ref. 9 at the anisotropy $\omega_x/\omega_0 = \sqrt{8}$ ($\omega_x = \omega_y = \omega_0$), in which work the authors have visualized the longtime dynamics by appropriately chosen Poincaré cuts. Here we summarize the most striking features of the phase-space structures at fixed L_z and E.

The Thomas–Fermi surface $\mu = U(\mathbf{r})$ is an axially symmetric ellipsoid. Important, classical characteristic energies of the problem are the chemical potencial μ and the centrifugal energy $\omega_0 L_z$. Let us introduce cylindrical coordinates $\rho = \sqrt{x^2 + y^2}$, z, ϕ. ϕ is a cyclic variable because of axial symmetry around the z axes. The Hamiltonian (12) has merely two degrees of freedom ρ and z, L_z enters only as a parameter.

If the energy and L_z is such that $E > (\omega_0 L_z)^2/4\mu > \mu$ two different kinds of trajectories can occur typically. If the the centrifugal energy $\omega_0 L_z$ is strong enough the particle cannot enter the condensate and is moving in a purely harmonic potential of the trap. This motion is completely integrable and as a third conserved quantity one can chose $E_z = p_z^2/2m + m\omega_z^2 z^2/2$. These trajectories show up in the Poincaré cut as integrable tori.

If the particle enters the condensate E_z is no longer conserved. Nevertheless, for energies $E \gg \mu$ trajectories are very similar to that of the motion in a harmonic trap (trivial Hartree–Fock limit). The influence of the condensate can be taken as a small perturbation in the external potential and the system behaves quasiintegrably. Trajectories are confined to thin stochastic layers separated from each other by integrable tori.

For energies comparable to the chemical potential one typically observes mixed phase space. The detailed phase space structure depends on the parameters chosen. Already a small anisotropy in the trap frequencies can lead to large chaotic parts in Poincaré cuts in this energy region. This indicates that for $E \approx \mu$ chaos is typical. This fact has been used in ref. 15, where in calculating the damping of elementary exciations the authors applied methods of random matrix theory based on the chaoticity of this very important classical region (the main contribution to the damping comes from those elementary excitation, where $E \approx \mu$).

If the energy is smaller than the chemical potential all trajectories move inside and outside the condensate. Decreasing the energy the chaotic part of phase space decreases and is restricted to a thin layer separating and surrounding two regular islands. Most orbits seem to lie in integrable tori for very small energy. This implies that the system has an integrable regime in the limit of small energies. Actually, there are two different kinds of integrable $E \ll \mu$ regime. One can be reached by the limit E/μ, $\omega_0 L_z/\mu \to 0$, keeping the ratio $\omega_0 L_z/E$ fixed. This limit, in a bulk case ($U(\mathbf{r}) = 0$) corresponds to the phonon regime, where the excitations show linear wave-number dependence. In the trapped case collective excitations in this regime has been studied first by Stringari.[11] The other low-energy integrable regime can be reached by the limit $E/\mu \to 0$ and lies where $E - (\omega_0 L_z)^2/4\mu \ll E$. Here the quasiparticles are single-particle-like excitations confined to a narrow layer around the Thomas–Fermi surface and described by the classical Hartree–Fock Hamiltonian. We call this regime by the nontrivial Hartree–Fock regime. In the follwing we shall investigate the above integrable regimes separately in more details.

4. INTEGRABLE LIMITING CASES CLASSICALLY

4.1. First Low-Lying Regime

Numerically it has been found[9] that tending with the energy to zero keeping μ, $E/\omega_0 L_z$ fixed the range of the classical motion outside the condensate for starting trajectories inside is getting smaller and smaller and in the limit the motion is confined to the region inside the Thomas–Fermi surface. Starting trajectories from the same point inside the condensate under the same direction and changing only the modulus of the Cartesian momentum there exist a well defined limiting orbit. Consequently, there should exist a well defined limiting Hamilton function. Inside the condensate the Bogoliubov Hamiltonian (12) can be written as

$$H(\mathbf{p},\mathbf{r}) = \sqrt{\frac{\mathbf{p}\cdot\mathbf{p}}{2m}\left(\frac{\mathbf{p}\cdot\mathbf{p}}{2m} + 2\mu - 2U(\mathbf{r})\right)} \tag{13}$$

For small energies $\mu - U(\mathbf{r})$ is much bigger than the kinetic energy everywhere except in a small region around the Thomas–Fermi surface, thus the approximant of the Bogoliubov Hamiltonian can be obtained by neglecting the kinetic energy square under the square root

$$H(\mathbf{p},\mathbf{r}) \simeq \sqrt{\frac{\mathbf{p}\cdot\mathbf{p}}{m}(\mu - U(\mathbf{r}))}, \qquad U(\mathbf{r}) < \mu \tag{14}$$

The Hamiltonian (14) has several interesting properties. It is a homogeneous first-order function of the momenta. From this fact follows that starting orbits from the same $\mathbf{r}(0)$ point with the same direction of momenta, but with different modulus $\mathbf{r}(t)$ is the same. Furthermore a constraint follows from the canonical equations: $m\mathbf{v}\cdot\mathbf{v} = \mu - U(\mathbf{r})$, relating the velocities and the coordinates i.e., they cannot be chosen independently. Third, for orbits approaching the boundary of the condensate the momentum diverges, the velocity vanishes at the Thomas–Fermi surface.

One can get some insight into the semiclassical trajectories in this regime if one realizes that sound wave motion with a spatially varying speed of sound is analogous to light propagation within an optical fiber. Here, the index of refraction is inversely proportional to the density of the condensate. Thus, the particles tend to be guided along the outer edge of the trap where the "index of refraction" is the largest.

The most important property of the Hamiltonian (14) is that the corresponding Hamilton–Jacobi equation is separable for *any* 3D harmonic oscillator trap potential.[9] For example, if the trap potential is isotropic one can use spherical coordinates. To prove separability in the experimentally most relevant axially symmetric trap potential case the separating variables are cylindrical elliptic coordinates ξ, η, ϕ defined as

$$\rho = \sigma_1\sqrt{(\xi^2+1)(1-\eta^2)}, \qquad z = \sigma_1\xi\eta, \quad \omega_z > \omega_0 \tag{15}$$

and

$$\rho = \sigma_2\sqrt{(\xi^2-1)(1-\eta^2)}, \qquad z = \sigma_2\xi\eta, \quad \omega_z < \omega_0 \tag{16}$$

depending on the ratio ω_z/ω_0, where

$$\sigma_1 = \sqrt{a^2 - b^2}, \quad \sigma_2 = \sqrt{b^2 - a^2}, \quad a = \sqrt{2\mu/m\omega_0^2}, \quad b = \sqrt{2\mu/m\omega_z^2} \tag{17}$$

The geometrical meaning of a and b are simply the semi axes of the Thomas–Fermi ellipsoid.

Following the separation procedure of ref. 9 one can find a new separation constant in addition to E and L_z, which gives a third classical conserved quantity $B(\mathbf{p}, \mathbf{r})$ which reads as

$$B = B(\mathbf{p}, \mathbf{r}) = a^2(p_x^2 + p_y^2) + b^2 p_z^2 - (xp_x + yp_y + zp_z)^2 \tag{18}$$

This nontvivial phase-space quantity ensures the integrability of this low-energy regime. We note that the separation procedure of the Hamilton–Jacobi equation for completely anisotropic harmonic potential can be found in refs. 9 and 16.

4.2. Nontrivial Hartree–Fock Regime

We have seen that the large energy (trivial) Hartree–Fock regime is quasi-integrable. In general, the motion given by the Hartree–Fock Hamiltonian H_{HF}

$$H_{HF}(\mathbf{p}, \mathbf{r}) \simeq \frac{p^2}{2m} + |\mu - U(\mathbf{r})| \tag{19}$$

is nonintegrable.[(9)]

However, in traps there is even a regime for energies much smaller than μ where on one hand the Hartree–Fock Hamiltonian (19) is a good approximation of the Bogoliubov–Hamiltonian and on the other hand the motion is integrable. Namely, in case when the kinetic energy is large compared to the local mean interaction energy $|\mu - U(\mathbf{r})|$ but the total energy E is still such that $E \ll \mu$. This can be satisfied in a layer around the Thomas–Fermi surface. In that case roughly speaking, the particle spends the same time inside and outside the condensate but the oscillations in the direction orthogonal to the Thomas–Fermi surface are much faster then the motion along the Thomas–Fermi surface. Consequently, there exists an adiabatic constant $I_\xi = (2\pi)^{-1} \oint p_\xi \, d\xi$ (integration over one full cycle in ξ) which emerges in this low-energy limit of the Hartree–Fock dynamics and which causes the integrability.

This adiabatic constant (for the case $\omega_0/\omega_z < 1$) in elliptic coordinates is given by[9]

$$I_\xi = \frac{4\mu}{3\pi\omega_z} \frac{1}{\sqrt{1-\varepsilon^2(1-\eta^2)}} \left[\frac{E}{\mu} - \frac{1-\eta^2}{1-\varepsilon^2(1-\eta^2)} \left(\frac{\omega_0 p_\eta}{2\mu} \right)^2 - \frac{1}{1-\eta^2} \left(\frac{\omega_0 L_z}{2\mu} \right)^2 \right]^{3/2} \tag{20}$$

where $\varepsilon = \sqrt{1-\omega_0^2/\omega_z^2}$. It is interesting to note that in this regime quasiparticles act as atoms in a "Mexican hat" potential, where the energy minimum is at the edge of the condensate.

5. COLLECTIVE LOW-LYING EXCITATIONS

The wave-equation for the low-lying exciations of the condensate had been obtained by Stringari.[11] For Bose–Einstein condensates at zero temperature, which are sufficiently large to validate the Thomas–Fermi approximation (6) he had found that one must solve the eigenvalue problem

$$\hbar^2\omega^2\Psi(\mathbf{r}) = \hat{G}\Psi(\mathbf{r}), \qquad \hat{G} = -\frac{\hbar^2}{m} \nabla(\mu - U(\mathbf{r}))\, \Theta(\mu - U(\mathbf{r}))\, \nabla \tag{21}$$

for the excitation frequencies ω and for the mode functions $\Psi(\mathbf{r})$ (for the connections of $\Psi(\mathbf{r})$ and the Bogoliubov amplitudes $u_j(\mathbf{r})$, $v_j(\mathbf{r})$ see ref. 17). In the semiclassical limit $\hbar\omega \to E$ and $-i\hbar\nabla \to \mathbf{p}$ the hydrodynamical equation (21) goes to the expression of the classical Hamiltonian (14). Thus, Eq. (21) is the quantized version of (14) with some prescribed ordering between the non-commuting operators $-i\hbar\nabla$ and $U(\mathbf{r})$. The wave equation must be solved with the normalization

$$\delta_{ij} = \int_{V_{TF}} d^3r\, \Psi_i^*(\mathbf{r})\, \Psi_j(\mathbf{r}) \tag{22}$$

(Integration is over the inside of the Thomas–Fermi surface). The eigenvalue problem (21) with the normalization (22) has discrete eigenmodes ω. This fact originates from the confining harmonical oscillator potential $U(\mathbf{r})$.

In the previous section we have analyzed the classical Hamiltonian problem and have seen that classical Hamilton–Jacobi equation was separable for harmonic traps. In the following we shall concentrate on the wave-equation (21) for different harmonic oscillator trap potential. The classical problem gives hints what coordinates might be good candidates in solving the wave-equation.

For isotropic traps ($\omega_x = \omega_y = \omega_z = \omega_0$) the angular momentum operators $\hat{L}_z$ and $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_x^2$ are commuting with $\hat{G}$, consequently l and m are good quantum numbers and trying the separation with $\Psi(\mathbf{r}) = r^l \Psi_n(r^2)\, Y_m^l(\theta, \varphi)$ we are left only with the radial problem. The normalization dictates that $\Psi_n(r^2)$ must be a finite polinomial of order n. Solving the eigenvalue problem Stringari showed[11] that the excitation spectrum is

$$\omega^2(n, l, m) = (2n^2 + 2nl + 3n + l) \tag{23}$$

He also had found some of the modes for axially symmetric Harmonic traps. In subsequent works[10, 18] the complete solution for the axially symmetric case was given. The method for solving the roost general $\omega_x \neq \omega_y \neq \omega_z$, triaxially anisotrqpic case can be found in ref. 16. Here we follow[10] in treating the $\omega_x = \omega_y = \omega_0$ case.

Because L_z is commuting with $\hat{G}$, m is a good quantum number. Let us investigate first when $\omega_0 < \omega_z$. Using cylindrical elliptical coordinates defined in (15) and the separation ansatz $\Psi(\xi, \eta, \varphi) = \Psi_\xi(\xi)\, \Psi_\eta(\eta)\, e^{im\varphi}$ one obtains two equations from (21): one for Ψ_η

$$\left[\frac{d}{d\eta}(1-\eta^2)\frac{d}{d\eta} - \frac{m^2}{1-\eta^2} + \frac{2\varepsilon^2(1-\eta^2)\,\eta}{1-\varepsilon^2(1-\eta^2)}\frac{d}{d\eta} - \frac{2\omega^2/\omega_0^2}{1-\varepsilon^2(1-\eta^2)}\right]\Psi_\eta = -\beta\Psi_\eta \tag{24}$$

the other for Ψ_ξ. β is a separation constant. It turns out that both equations are identical if in the equation for Ψ_η we substitute $i\xi$ for η, i.e., $\Psi_\xi(\xi) = \Psi_\eta(i\xi)$. The solution for one coordinate is the analytical continuation of the solution of the other from the real to the imaginary axis. Equation (24) depends only on m^2 not on m, therefore the energy levels are the same for $\pm m$. Expanding Ψ_η for fixed m in terms of associated Legendre function $P_l^{|m|}(\eta)$ with arbitrary coefficients a_l, (where $|m| \leqslant l$) one obtains a second order recursion relation for the coefficients a_l relating only odd or even indices l. The scalar product (22) and the connection between $\Psi_\xi(\xi)$ and $\Psi_\eta(i\xi)$ require that this second order recursion must terminate at some integer $l_{\max} = |m| + n$. This leads to the quantization of the the separation constant β:

$$\beta = (n + |m|)(n + |m| + 3) \tag{25}$$

and to the fact that Ψ_η (and correspondingly Ψ_ξ) must be $(1-\eta^2)^{|m|/2}$ times a polinomial of order n. The eigenvalue condition for the excitation frequencies in the bases chosen takes the form of a tridiagonal matrix, which can be symmetrized, and therefore has real eigenvalues. The dimension of this matrix is $N = 1 + [n/2]$, and for fixed n, $|m|$ we have N different solutions for ω^2, which we label by the third quantum number $j = 0, \ldots, N-1$. If $N = 1$ or $N = 2$ one can diagonalize the matrix by hand. Energy levels for these cases can be found in ref. 10.

One can construct such an operator $\hat{B}$, which acting on the wave-function has the eigenvalue β. This operator in cartesian coordinates has the form

$$\hat{B} = (\mathbf{r}\nabla)(\mathbf{r}\nabla + 3) - a^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) - b^2 \frac{\partial^2}{\partial x_3^2} \tag{26}$$

and is a quantized version of the classically conserved quantity B (see Eq. (18)). The operatror $\hat{B}$ commutes with $\hat{G}$ and $\hat{L}_z$ in the axially symmetric case and ensures the complete separability of Eq. (21). This operator is hermitian with respect to the scalar product used.

For a general anisotropy ε analytical results can be obtained only for $N = 1$ and $N = 2$. But, for two special values of ε full analytic results are possible. One of them is $\varepsilon = 0$, which is the spherical symmetric case and the case $\varepsilon = 1$, which was partially investigated by Stringari in ref. 19. As a new result here, we reconsider the $\varepsilon = 1$ case using the above approach to the problem. $\varepsilon \to 1$ is an interesting, but rather singular limit (no experimental realization till now). Keeping ω_z at finite value and tending with ω_0 to zero is a possible realization of this limit. In that case the Thomas–Fermi condensate deforms to a more and more extended pancake-like object and at $\varepsilon = 1$ it extends to infinity. However, the excitation spectrum tends to a well defined limiting spectrum. Using as a small parameter $\delta = 1 - \varepsilon^2$ in perturbation theory one can obtain analytic results for highly deformed trap. Let us intruduce the following notations:

$$\frac{\omega^2(\delta)}{\omega_z^2} = \lambda(\delta) \tag{27}$$

Multiplying both sides of Eq. (24) by $1 - \varepsilon^2(1 - \eta^2)$ and inserting β from (25) the equation to be solved has the form $\lambda(\delta)\, \Psi_\eta = (\hat{H}_0 + \delta \hat{H}_1)\, \Psi_\eta$. Writing

$$2\lambda(0) = (n - 2j)(n - 2j + 1), \qquad \Psi_\eta(\eta) = (1 - \eta^2)^{|m|/2}\, \eta^{n-2j} \tilde{\Psi}(\eta^2) \tag{28}$$

(with j undetermined till now) one gets a hipergeometric equation for $\tilde{\Psi}(x)$, $x = \eta^2$. Due to the scalar product (22) the only possible solution is

$$\Psi_\eta(\eta, \delta = 0) = (1 - \eta^2)^{|m|/2}\, \eta^{n-2j} F(-j, n + |m| - j + 3/2; n - 2j + 3/2, \eta^2),$$
$$j = 0, 1, \ldots, [n/2] \tag{29}$$

Thus, at $\delta=0$ j is our third quantum-number introduced above. It is important to note that the spectrum (28) at $\delta=0$ do not depend on $|m|$. After a rather tedious, but straighforward calculation one obtains for the spectrum:

$$\omega^2=\omega_z^2\left[\frac{(n-2j)(n-2j+1)}{2}+\frac{\omega_0^2}{\omega_z^2}\left(|m|+\frac{2j(n+|m|-j+3/2)}{2j-n-3/2}\right)+o\left(\frac{\omega_0^0}{\omega_z^4}\right)\right] \tag{30}$$

using first order perturbation theory. If $\omega_z \gg \omega_0$ the spectrum has a large scale ω_z plus a fine structure on a much smaller scale. If $n=2p$, $j=p$ ($p=$ 0, 1,...) the large scale part vanishes. In that case the spectrum reduces to

$$\omega^2 \approx \omega_0^2(|m|+\tfrac{4}{3}p\,|m|+\tfrac{4}{3}p^2+2p) \tag{31}$$

Those soft modes have two integer quantum numbers and appearing on a two-dimensional manifold of the (n, j, m) quantum-number space. The soft modes (31) have been first calculated by Stringari[19] but, not the full spectra (30).

Now, let us turn to the opposite case $\omega_0 > \omega_z$. One must use the coordinates (16). The treatment is very similar to the previous calculation. Here we only summarize the differences. The separation ansatz is the same as before, but (24) is replaced by

$$\left[\frac{d}{d\eta}(1-\eta^2)\frac{d}{d\eta}-\frac{m^2}{1-\eta^2}+\frac{2\tilde{\varepsilon}^2(1-\eta^2)\,\eta}{(1-\tilde{\varepsilon}^2\eta^2)}\frac{d}{d\eta}-\frac{2\omega^2/\omega_0^2}{1-\tilde{\varepsilon}^2\eta^2)}\right]\Psi_\eta=-\beta\Psi_\eta \tag{32}$$

where $\tilde{\varepsilon}^2=1-\omega_z^2/\omega_0^2$. The connection of Ψ_ξ and Ψ_η is $\Psi_\xi(\xi)=\Psi_\eta(\xi)$. Quantum numbers n, m, j have the same meanings and the possible values for the separation constant β are also given by (25). Matrixes are tridiagonal too, but have different matrix elements. Analytically solvable cases are $N=1$, $N=2$, and $\tilde{\varepsilon}=0, 1$. The $\tilde{\varepsilon}\approx 1$ has been experimentally realized[20] (in the experiment $\omega_z/\omega_0=17/230$ has been chosen) and the first $m=0$ excitation mode has been measured with very high precision in accordance with Stringari's result. Using perturbation theory direct calculation gives for the unnormalized solutions:

$$\Psi_\eta(\eta, \tilde{\delta}=0)=\text{Const}(1-\eta^2)^{-1/2}\,P_{n+|m|+1}^{|m|+2j+1}(\eta) \tag{33}$$

where $\tilde{\delta}=1-\tilde{\varepsilon}^2=\omega_z^2/\omega_0^2$ and $P_m^l(x)$ is the associated Legendre-function of the first kind. Results for excitations given by first order perturbation theory for small $\tilde{\delta}$ are

$$\omega^2(n, m, j=0) = \omega_0^2 \left[|m| + \left(\frac{\omega_z^2}{\omega_0^2}\right) \frac{n(n+2|m|+3)}{2(|m|+2)} + o\left(\frac{\omega_z^4}{\omega_0^4}\right) \right] \tag{34}$$

$$\begin{aligned} \omega^2(n, m, j\neq 0) = \omega_0^2 \Bigg\{ & [|m|(2j+1) + j(2j+2)] + \left(\frac{\omega_z^2}{\omega_0^2}\right) \\ & \times \left[\frac{(n-2j)(n+2|m|+2j+3)}{2(|m|+2j+2)} + \frac{j(j+|m|)}{2} \right. \\ & \times \left(\frac{(n-2j)(n+2|m|+2j+3)}{(|m|+2j+1)(|m|+2j+2)} \right. \\ & \left.\left. + \frac{(n-2j+1)(n+2|m|+2j+2)}{(|m|+2j+1)(|m|+2j)} \right) \right] + o\left(\frac{\omega_z^4}{\omega_0^4}\right) \Bigg\} \end{aligned} \tag{35}$$

$j = 0, 1,..., [n/2]$. This complete result is in accordance with Stringari,[19] who has calculated only the dispersion relation of the $m=0$, $j=0$ soft modes for $\omega_z \ll \omega_0$:

$$\omega^2(n, m=0, j=0) \approx \omega_z^2 \frac{n(n+3)}{4} \tag{36}$$

using a different approach to the problem.

6. FINAL REMARKS

We have seen that in the framework of the Bogoliubov theory chaos can show up if the condensate is taken by the Thomas–Fermi approximation. This condensate wave-function is nonanalytic at the edge of the condensate and may affect the existence of chaos. Taken a smooth condensate given by the solution of the Gross–Pitaevskii equation chaos still shows up, which is supported by the numerical work of ref. 21. In the spectra avoided level crossings can be found, which are clear signs of quantum chaos. Thus the existence of chaos is not connected to the Thomas–Fermi approximation (which is rather good at large condensates).

The integrability of the non-trivial Hartree–Fock regime is however connected to the Thomas–Fermi approximation. This regime is hardly accessible analytically for smooth condensates. It might be possible that with smooth condensate wave-function this regime is not integrable at all.

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